ELSEVIER
BULLETIN
www.elsevier.com/locate/bulsci

Horizontal lift of Ornstein–Uhlenbeck process over Riemannian path space ☆

Xicheng Zhang

Department of Mathematics, Huazhong University of Science and Technology, Wuhan, Hubei 430074, PR China

Received 27 January 2004; accepted 2 February 2004

Available online 18 March 2004

Abstract

In this paper we give a representation formula for the heat semigroup on adapted vector fields by the horizontal lift of Ornstein–Uhlenbeck process over Riemannian path space.

MSC: 60H07; 60H20; 53C05

Keywords: Horizontal lift; Ornstein–Uhlenbeck process; Riemannian path space; Heat semigroup

1. Introduction

Let (M, g) be a compact Riemannian manifold with dimension d. Let us consider the following path space:

$$P(M) := \{p : [0,1] \to M,\ \text{continuous},\ p(0) = x_0\},$$

where $x_0 \in M$ is fixed. On $P(M)$ we endow the Wiener measure ν induced by the Brownian motion on M. The path space has a natural structure of Itô filtration and a differentiable structure which is transported from the Malliavin calculus on the (flat) Wiener space through the Itô map: Itô stochastic parallel transport provides a methodology of moving frames on the path space.

Cruzeiro and Malliavin [2] defined the Riemannian geometry on $P(M)$. Since the Itô map is not Cameron–Martin differentiable and the naturally defined Levi-Civita connection

---

☆ This work is supported by NSF (No. 10301011) of China and Project 973.
E-mail address: xczhang@hust.edu.cn (X. Zhang).

0007-4497/$ – see front matter
doi:10.1016/j.bulsci.2004.02.001

is a rather divergent object, a notion of Markovian connection was introduced in [2]. This connection is Riemannian but has a torsion. Computing the corresponding Weitzenbock formula gives rise to nontrivial first order terms which are hard to estimate.

In [3] a renormalization procedure was defined meaning that identities such as the Weitzenbock formula are simplified when we restrict them to adapted vector fields. The underlying principle is that differential geometry on the path space should be compatible with the Itô filtration. In particular, the bundle of orthogonal frames to the path space is restricted to those which commute with the projections determined by the conditional expectation with respect to the Itô filtration.

The construction of Ornstein–Uhlenbeck process on $P(M)$ was first given by Driver and Röckner [6] by the quasi regular Dirichlet form. Later, Kazumi [7] gave another construction by projection technique. Norris [8] applied two parameter stochastic calculus to construct the process on $P(M)$. It is worthy to point that Driver–Röckner process coincides with Norris process only when Ricci tensor vanishes.

In [5] a finite dimensional approximation of Riemannian path space geometry was constructed. In particular, this approximation allows to construct the horizontal lift of the Ornstein–Uhlenbeck process on the path space through the Markovian connection. Then a representation formula for the heat semigroup on adapted vector fields was given by this lift. However, this lift lives on a big space, the bundle of measurable orthogonal frames. Here we shall prove this representation by the process constructed by Norris [8] which lives on the bundle of continuous orthogonal frames. The lifted process plays a crucial role in the development of the stochastic calculus of variations on the path space [4].

2. Main results and proofs

In this paper we shall keep all the notations and notions in Norris [8]. We consider the Levi-Civita connection ∇ on tangent bundle TM. Let x_{st} be a semimartingale on M in the sense of Norris [8]. The covariant Stratonovich differential along x_{st} is defined by

$$D_s := \partial_s + A(\partial_s x_{st}),$$

so that

$$D_s \partial_t x_{st} = \partial_s \partial_t + A(\partial_s x_{st}) \partial_t x_{st}$$

and for a process z_{st} in TM over x_{st} with appropriate semimartingale properties

$$D_s z_{st} = \partial_s z_{st} + A(\partial_s x_{st}) z_{st},$$
$$D_s D_t z_{st} = \big(\partial_s + A(\partial_s x_{st})\big)\big(\partial_t z_{st} + A(\partial_t x_{st}) z_{st}\big),$$

where $A_i \in End(\mathbb{R}^n)$ is locally given by $g(\nabla_{\partial x^i} \partial x^j, \partial x^k)$. Then we can easily check that

$$D_s \partial_t x_{st} = D_t \partial_s x_{st},$$
$$D_s D_t z_{st} = D_t D_s z_{st} + R(\partial_s x_{st}, \partial_t x_{st}) z_{st},$$

where Ω is the curvature tensor on M corresponding to ∇.

Norris [8, p. 330] proved the following result.

Theorem 1. *Consider the system of differential equation*

$$\begin{cases} D_s\partial_t x_{st} = u_{st}\partial_s\partial_t w_{st} - \frac{1}{2}\partial_s x_{st}\partial_t + \frac{1}{2}\mathrm{Ricci}(\partial_t x_{st})\partial_s, \\ D_s u_{st} = 0, \\ D_t v_{st} = 0, \end{cases}$$

subjected to the boundary condition

$$\begin{cases} x_{s0} \text{ is a given Brownian motion on } M \text{ starting from } m_0; \\ x_{0t} = m_0, u_{00} = v_{00} = \text{a given orthogonal frame over } m_0; \\ u_{0t} = v_{0t}, v_{s0} = u_{s0}. \end{cases}$$

Then there are unique continuous semimartingales $\{x_{st}, u_{st}, v_{st}\}$ *satisfying these equations. In particular,* $x_t := \{x_{st}, s \geqslant 0\}$ *is an Ornstein–Uhlenbeck process on path space* $P(M)$. *Moreover,* $s \mapsto u_{st}$ *is the horizontal lift of* $s \mapsto x_{st}$ *for fixed* t *by the Levi-Civita connection, and* $t \mapsto v_{st}$ *the horizontal lift of* $t \mapsto x_{st}$ *for fixed* s.

Let $P(O(d))$ denote the space of continuous mappings $e(s)$ from $[0,1]$ to $O(d)$, orthogonal matrix group, with $e(0) = I$. Define the following process taking values in $P(O(d))$

$$e_{st} := u_{st}^{-1} v_{st}.$$

The process $r_{\cdot t} := (x_{\cdot t}, e_{\cdot t})$ taking values in the frame bundle $O(P(M)) = P(M) \times P(O(d))$ will be called the horizontal lift of Ornstein–Uhlenbeck process x_t through Markovian connection on path space defined by Cruzeiro and Malliavin [2].

Proposition 2. *The process* r_t *has the following Markovian property*:

$$x_{t+t'}(x_0, w) = x_{t'}\big(x_t(x_0, w), \theta_t(w)\big),$$
$$e_{t+t'} \cdot e_t^{-1}(x_0, w) = e_{t'}\big(x_t(x_0, w), \theta_t(w)\big),$$

where $[\theta_t(w)]_{st'} := w_{s(t+t')} - w_{st}$.

Proof. Similar to Norris [8, p. 327], we set

$$\tilde{x}_{st'} = x_{s(t+t')},$$
$$\tilde{u}_{st'} = u_{s(t+t')},$$
$$\tilde{v}_{st'} = v_{s(t+t')} v_{st}^{-1} u_{st}.$$

Then $\tilde{x}_{st'}, \tilde{u}_{st'}, \tilde{v}_{st'}$ are determined by

$$\begin{cases} D_s\partial_{t'}\tilde{x}_{st'} = \tilde{u}_{st'}\partial_s\partial_{t'}[\theta_t(w)]_{st'} - \frac{1}{2}\partial_s\tilde{x}_{st'}\partial t' + \frac{1}{2}\mathrm{Ricci}(\partial_{t'} x_{st'})\partial s, \\ D_s\tilde{u}_{st'} = 0, \\ D_{t'}\tilde{v}_{st'} = 0 \end{cases}$$

and initial conditions $\tilde{x}_{s0} = x_{st}, \tilde{x}_{0t'} = x_0$ and $\tilde{u}_{s0} = \tilde{v}_{s0}, \tilde{u}_{0t'} = \tilde{v}_{0t'}$.

The uniqueness of the solution implies the desired Markovian property. □

Let $\mathcal{C}$ be the total of cylindrical functions on path space, H the Cameron–Martin space. We define $\mathcal{C}(H)$ as the set of cylindrical vector fields in $L^2(W(M),\nu;H)$, namely

$$\mathcal{C}(H):=\left\{X(p)=\sum_{i=1}^{k}F_i(p)h_i,\ F_i\in\mathcal{C},\ h_i\in H\right\}.$$

Define

$$T_t^H Z(x_0):=\mathbb{E}^{x_0}\big(e_t^{-1}Z(x_t)\big).$$

Then

Proposition 3. *T_t^H is a symmetric semigroup on $L^2(W(M),\nu;H)$.*

Proof. The symmetry follows from the reversibility of x_t(cf. Norris [8, p. 331]). Moreover, by the Markovian property of (x_t,e_t), we have

$$\begin{aligned}T_t^H T_s^H Z(x_0)&=\mathbb{E}^{x_0}\big(e_t^{-1}\mathbb{E}^{x_t}\big(e_s^{-1}Z(x_s)\big)\big)\\&=\mathbb{E}^{x_0}\big(e_t^{-1}\mathbb{E}^{x_0}\big(e_t e_{s+t}^{-1}Z(x_{s+t})\mid\mathcal{F}_t\big)\big)\\&=\mathbb{E}^{x_0}\big(e_{s+t}^{-1}Z(x_{s+t})\big)=T_{t+s}^H Z(x_0).\qquad\square\end{aligned}$$

For $Y,Z\in\mathcal{C}(H)$, the Markovian connection is defined by

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\nabla_Y^M Z:=D_Y\dot{Z}+Q_Y\dot{Z},$$

where $Q_Y(\tau):=\int_0^\tau\Omega(Y,\partial b)\in so(d)$, b is the d-dimensional standard Brownian motion in $\mathbb{R}^d$.

Let $\{h\}$ runs over an orthogonal normal bases of Cameron–Martin space H. Define the following closable symmetric bilinear form

$$\mathcal{E}^H(Z,Y):=\sum_h\mathbb{E}\big(\nabla_h^M Z\mid\nabla_h^M Y\big),\quad Z,Y\in\mathcal{C}(H).$$

Then we have (cf. [1])

Theorem 4. *The generator L^H of $(\mathcal{E}^H,\mathcal{C}(H))$ is given by*

$$\begin{aligned}L^H X&=\sum_h\big(-\nabla_h\nabla_h X+\delta(h)\nabla_h X\big)\\&=-\sum_\alpha\int_0^1\nabla_{\tau,\alpha}^2 X\,\mathrm{d}\tau+\sum_\alpha\int_0^1\nabla_{\tau,\alpha}X\circ\mathrm{d}x_\tau^\alpha.\end{aligned}$$

In particular, for $Z=Fz$ and $Y=Gy$, we have

$$\mathcal{E}^H(Z,Y)=\sum_h \mathbb{E}\big(D_hFD_hG(z,y)_H\big)$$
$$+\mathbb{E}\left(D_h(FG)\int_0^1 \dot{y}^T(\tau)Q_h(\tau)\dot{z}(\tau)\,\mathrm{d}\tau\right)$$
$$+\mathbb{E}\left(FG\int_0^1 \dot{y}^T(\tau)Q_h^T(\tau)Q_h(\tau)\dot{z}(\tau)\,\mathrm{d}\tau\right).$$

We can prove our main result.

Theorem 5. *For any $Z,Y\in\mathcal{C}(H)$, if Ricci tensor vanishes, it holds that*

$$\mathcal{E}^H(Z,Y)=\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}\mathbb{E}\Big(\big(e_t^{-1}Z(x_t)-Z(x_0)\mid e_t^{-1}Y(x_t)-Y(x_0)\big)_H\Big). \tag{1}$$

Proof. Without any loss of generality, we may assume $Z=Fz$, $Y=Gy$ by the bilinearity of $\mathcal{E}$. Since $e_t^{-1}Z(x_t)-Z(x_0)$ and $e_t^{-1}Y(x_t)-Y(x_0)$ are two semimartingales in H with initial value 0, we have

$$\mathbb{E}\Big(\big(e_t^{-1}Z(x_t)-Z(x_0)\mid e_t^{-1}Y(x_t)-Y(x_0)\big)_H\Big)=\mathbb{E}\big([e^{-1}Z,e^{-1}Y]_t^H\big)$$

where $[\cdot,\cdot]^H$ denotes the covariation in H.

First of all, let us calculate $\partial_s\partial_te_{st}^{-1}$.

$$\begin{aligned}
\partial_s\partial_te_{st}^{-1}&=\partial_s\partial_t\big(v_{st}^{-1}u_{st}\big)\\
&=\partial_s\big(\big(\partial_tv_{st}^{-1}\big)u_{st}+v_{st}^{-1}\partial_tu_{st}\big)\\
&=\partial_s\big(-v_{st}^{-1}(\partial_tv_{st})v_{st}^{-1}u_{st}+v_{st}^{-1}\partial_tu_{st}\big)\\
&=\partial_s\big(v_{st}^{-1}D_tu_{st}\big)\\
&=v_{st}^{-1}D_sD_tu_{st}-v_{st}^{-1}D_sv_{st}v_{st}^{-1}D_tu_{st}\\
&=v_{st}^{-1}\big(D_tD_su_{st}-R(\partial_tx_{st},\partial_sx_{st})u_{st}\big)-v_{st}^{-1}D_sv_{st}v_{st}^{-1}D_tu_{st}\\
&=v_{st}^{-1}R(\partial_sx_{st},\partial_tx_{st})u_{st}-v_{st}^{-1}D_sv_{st}v_{st}^{-1}D_tu_{st}.
\end{aligned}$$

Therefore

$$\partial_te_{st}^{-1}=\int_0^s v_{rt}^{-1}R(\partial_rx_{rt},\partial_tx_{rt})u_{rt}-\int_0^s v_{rt}^{-1}D_rv_{rt}\cdot v_{rt}^{-1}D_tu_{rt}=:\partial_tJ_{st}^1+\partial_tJ_{st}^2.$$

Now

$$\begin{aligned}
&\partial_t\big((e_t^{-1}z\mid h)_HF(x_t)\big)\partial_t\big((e_t^{-1}y\mid h)_HG(x_t)\big)\\
&\quad=\big((\partial_te_t^{-1}z\mid h)_HF(x_t)+(e_t^{-1}z\mid h)_H\partial_tF(x_t)\big)\\
&\qquad\times\big((\partial_t^{-1}e_ty\mid h)_HG(x_t)+(e_t^{-1}y\mid h)_H\partial_tG(x_t)\big)\\
&\quad=:I_1(t)+I_2(t)+I_3(t)+I_4(t),
\end{aligned}$$

where

$$\begin{aligned}
I_1(t) &:= \big(\partial_t e_t^{-1} z \mid h\big)_H F(x_t)\big(\partial_t e_t^{-1} y \mid h\big)_H G(x_t),\\
I_2(t) &:= \big(\partial_t e_t^{-1} z \mid h\big)_H F(x_t)\big(e_t^{-1} y \mid h\big)_H \partial_t G(x_t),\\
I_3(t) &:= \big(e_t^{-1} z \mid h\big)_H \partial_t F(x_t)\big(\partial_t e_t^{-1} y \mid h\big)_H G(x_t),\\
I_4(t) &:= \big(e_t^{-1} z \mid h\big)_H \partial_t F(x_t)\big(e_t^{-1} y \mid h\big)_H \partial_t G(x_t).
\end{aligned}$$

From the calculation of Norris we have (cf. [8, p. 333])

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}\mathbb{E}\big(I_4(t)\big) = \sum_h \mathbb{E}\big(D_h F \cdot D_h G(z, y)_H\big).$$

For $I_1(t)$, by the following formula (cf. [8, p. 333])

$$\partial_t x_{st} \otimes \partial_t x_{s't} = \sum_h (u_{st} h_s \otimes u_{s't} h_{s'})\partial_t,$$

we have

$$\partial_t e_t^{-1} \otimes \partial_t e_t^{-1} = \partial_t J_{st}^1 \otimes \partial_t J_{st}^1 + \partial_t J_{st}^1 \otimes \partial_t J_{st}^2 + \partial_t J_{st}^2 \otimes \partial_t J_{st}^1 + \partial_t J_{st}^2 \otimes \partial_t J_{st}^2.$$

Note that

$$\begin{aligned}
\partial_t J_{st}^1 \otimes \partial_t J_{st}^1 &= \int_0^s \int_0^s v_{rt}^{-1} R^T(\partial_r x_{rt}, \partial_t x_{rt}) u_{rt} u_{r't}^{-1} R(\partial_{r'} x_{r't}, \partial_t x_{r't}) v_{r't}\\
&= \sum_h \left(\int_0^s \int_0^s v_{rt}^{-1} R^T(\partial_r x_{rt}, u_{rt} h_r) R(\partial_{r'} x_{r't}, u_{r't} h_{r'}) v_{r't}\right)\partial t
\end{aligned}$$

and

$$\begin{aligned}
\partial_t J_{st}^1 \otimes \partial_t J_{st}^2\big|_{t=0} &= 0,\\
\partial_t J_{st}^2 \otimes \partial_t J_{st}^1\big|_{t=0} &= 0,\\
\partial_t J_{st}^2 \otimes \partial_t J_{st}^2\big|_{t=0} &= 0.
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}\mathbb{E}\big(I_1(t)\big) &= \sum_h \mathbb{E}\Bigg(FG \int_0^1 \dot{y}^T(s)\Bigg(\int_0^s \int_0^s v_{r0}^{-1} R^T(\partial_r x_{r0}, u_{r0} h_r)\\
&\quad \times R(\partial_{r'} x_{r'0}, u_{r'0} h_{r'}) v_{r'0}\Bigg)\dot{z}(s)\,\mathrm{d}s\Bigg)\\
&= \sum_h \mathbb{E}\Bigg(FG \int_0^1 \dot{y}^T(s)\Bigg(\int_0^s \Omega^T(\partial b_r, h_r) \int_0^s \Omega(\partial b_{r'}, h_{r'})\Bigg)\dot{z}(s)\,\mathrm{d}s\Bigg)
\end{aligned}$$

$$= \sum_h \mathbb{E}\left(FG \int_0^1 \dot{y}^T(s) Q_h^T(s) Q_h(s) \dot{z}(s)\, \mathrm{d}s \right),$$

where we have used that $u_{r0}^{-1} \partial_r x_{r0} = \partial b_r$.

Lastly, let F have the following form

$$F(x) = f(x_{s_1}, \ldots, x_{s_n}).$$

By differential formula we have

$$\partial_t F(x_t) = \sum_{i=1}^n d_i f(\partial_t x_{s_i t}).$$

Thus

$$\begin{aligned}
\partial_t F(x_t) \partial_t e_t^{-1} &= \sum_{i=1}^n d_i f(\partial_t x_{s_i t}) \otimes \left(\partial_t J_{st}^1 + \partial_t J_{st}^2\right) \\
&= \left(\sum_h \sum_{i=1}^n d_i f(u_{s_i t} h_{s_i}) \int_0^s R(\partial_r x_{rt}, u_{rt} h_r) v_{rt} \right) \partial t \\
&\quad + \sum_{i=1}^n d_i f(\partial_t x_{s_i t}) \otimes \partial_t J_{st}^2,
\end{aligned}$$

and by $\partial_t x_{s_i t} \otimes \partial_t J_{st}^2|_{t=0} = 0$ we have

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathbb{E}\big(I_3(t)\big) = \mathbb{E}\left(G D_h F \int_0^1 \dot{y}^T(\tau) Q_h(\tau) \dot{z}(\tau)\, \mathrm{d}\tau \right).$$

Similarly for $I_2(t)$ and we complete the proof. $\square$

Now we prove that the generator of T_t^H is just given by L^H.

Theorem 6. *For any $Z \in \mathcal{C}(H)$,*

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} T_t^H Z = -\frac{1}{2} L^H Z.$$

Proof. We only need to prove that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathbb{E}\big((T_t^H Z \mid Y)_H\big) = -\frac{1}{2} \mathbb{E}\big((L^H Z \mid Y)_H\big)$$

for any $Y \in \mathcal{C}(H)$. Note that $\mathbb{E}((e_t^{-1} Z(x_t) \mid e_t^{-1} Y(x_t))_H) = \mathbb{E}((Z(x_t) \mid Y(x_t))_H) = \mathbb{E}((Z \mid Y)_H)$, by the symmetry of T_t^H, the right hand of (1) is equal to

$$-\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}\mathbb{E}\big((T_t^H Z \mid Y)_H\big) - \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}\mathbb{E}\big((Z \mid T_t^H Y)_H\big)$$
$$= -2\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}\mathbb{E}\big((T_t^H Z \mid Y)_H\big) = \mathbb{E}\big((L^H Z \mid Y)_H\big),$$

and which gives the proof. □

Acknowledgement

The author is very grateful to Professors Ana-Bela Cruzeiro and Jiagang Ren for their very helpful discussions and suggestions.

References

[1] A.B. Cruzeiro, S. Fang, Weak Levi-Civita connection for the damped metricon the Riemannian path space and vanishing of Ricci tensor in adapted differential geometry, J. Funct. Anal. 185 (2001) 681–698.
[2] A.B. Cruzeiro, P. Malliavin, Renormalized differential geometry on path space: structural equation, curvature, J. Funct. Anal. 139 (1996) 119–181.
[3] A.B. Cruzeiro, P. Malliavin, Frame bundle of Riemannian path space and Ricci tensor in adapted differential geometry, J. Funct. Anal. 177 (2000) 219–253.
[4] A.B. Cruzeiro, P. Malliavin, Stochastic calculus of variations and Harnack inequality on Riemannian path space, C.R. Acad. Sci. Paris, Sér. I 335 (2002) 817–820.
[5] A.B. Cruzeiro, X. Zhang, Finite dimensional approximation of Riemannian path space geometry, J. Funct. Anal. 205 (2003) 206–270.
[6] B.K. Driver, M. Röckner, Construction of diffusions on path and loop spaces of compact Riemannian manifolds, C.R. Acad. Sci. Paris, Sér. I 315 (1992) 603–608.
[7] T. Kazumi, Les processes d'Ornstein–Uhlenbeck sur l'espace de chemins Riemanniens et le problem des martingales, J. Funct. Anal. 144 (1997) 20–45.
[8] J.R. Norris, Twisted sheets, J. Funct. Anal. 132 (1995) 273–334.